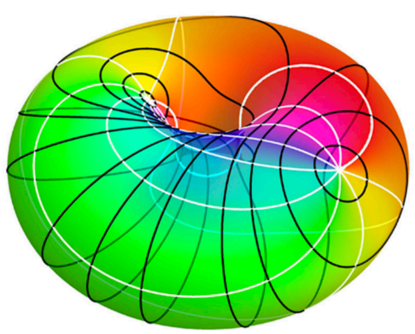
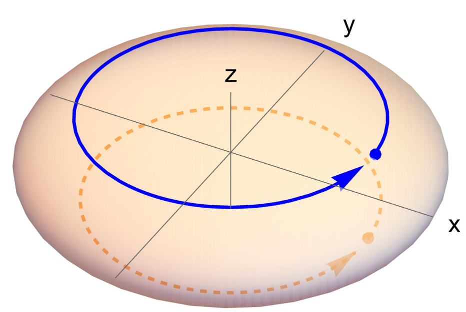
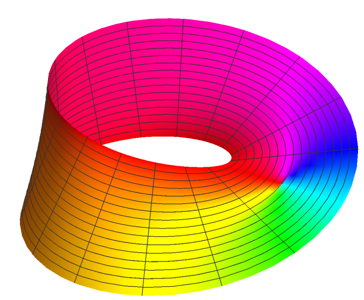
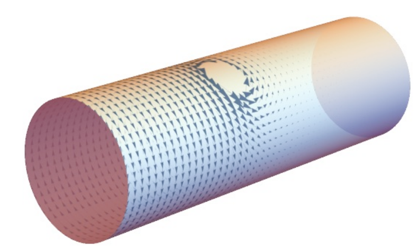
Superfluid vortex dynamics on peculiar surfaces

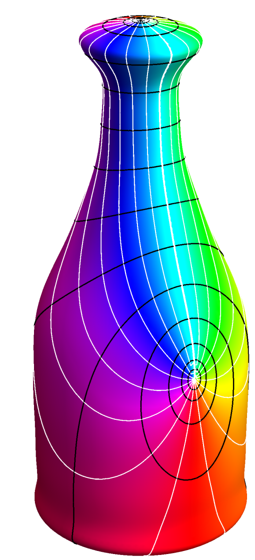
Pietro Massignan











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Outline

- Classical vs. superfluid turbulence
- Perfect fluids in 2D
- Vortices on the plane
- Conformal mappings and isothermal coordinates
- Vortices on cylinders and torii won’t stand still
- Cones, Möbius strips, airplane wings, champagne bottles, …

Collaborators

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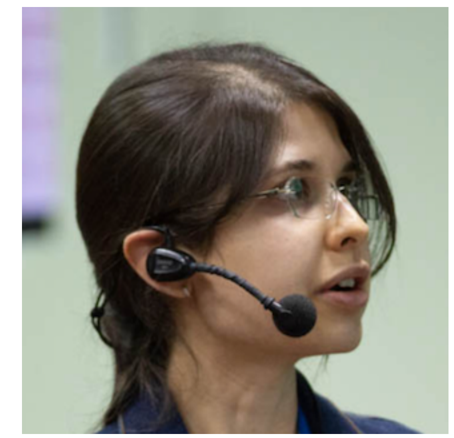
Alexander Fetter

LELAND STANFORD JUNIOR UNIVERSITY
DIE LUFT DER FREIHEIT WEHT
1891



Mônica Caracanhas

USP
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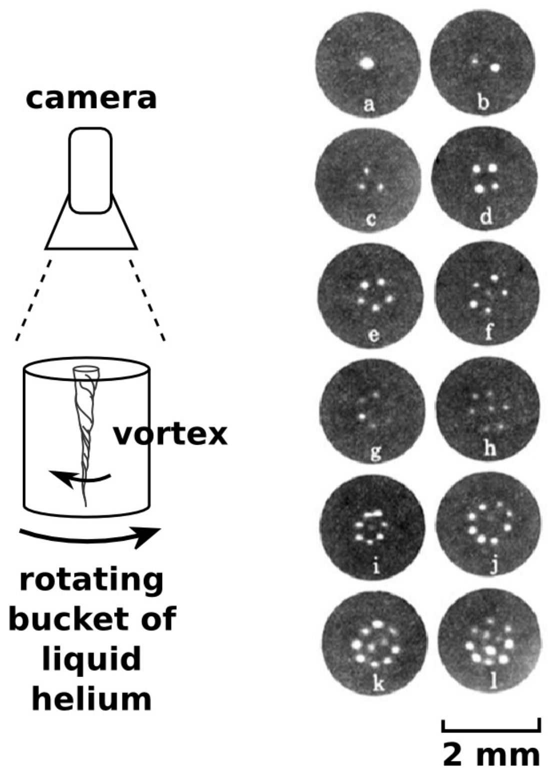
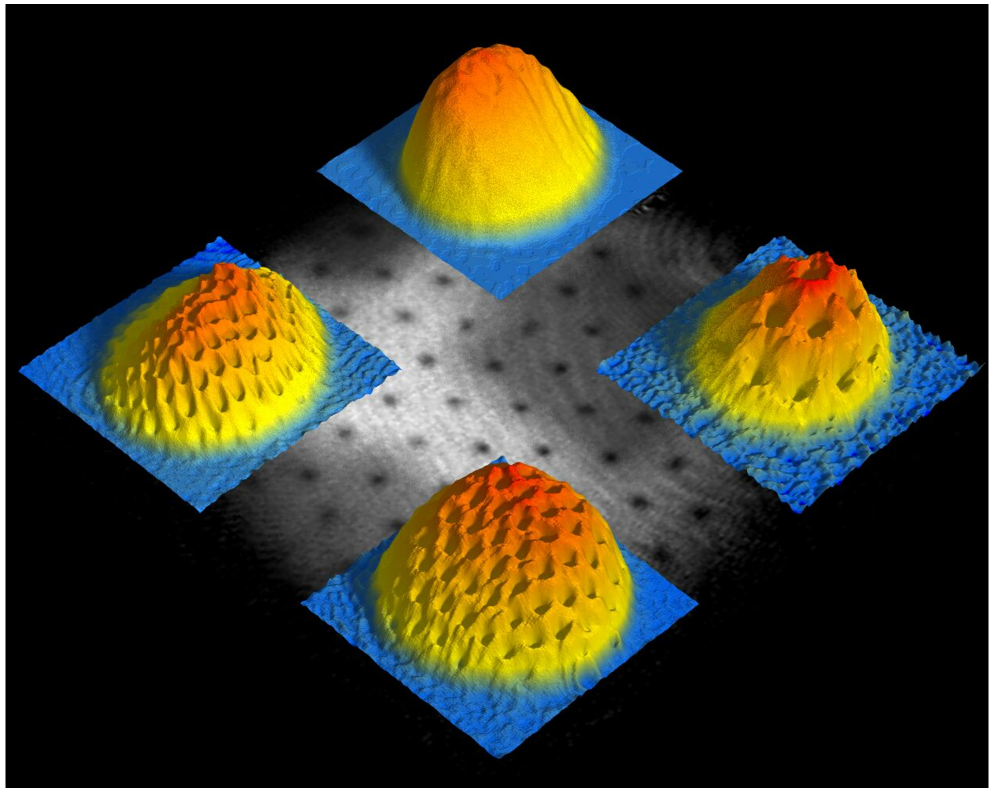
Classical turbulence

viscous
multiscale
chaotic

Superfluid vortices



[Yarmchuk, Gordon and Packard, 1979]



[Ketterle's group @ MIT, 2001]

no viscosity
quantized circulation
ordered vortex patterns

Superfluid hydrodynamics

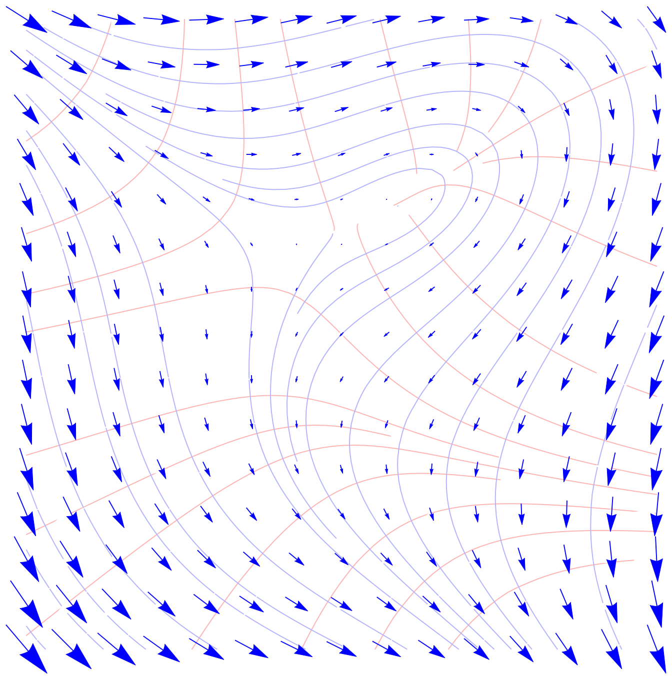
- Macroscopic condensate wavefunction: $\Psi = \sqrt{n} e^{i\Phi}$
- Superfluid velocity: $\mathbf{v} = \frac{\hbar}{M} \nabla \Phi$
- Vorticity: $\nabla \times \mathbf{v} = \frac{\hbar}{M} \nabla \times \nabla \Phi = 0$ (*irrotational*)
- Quantized circulation: $\oint \mathrm{d}\mathbf{l} \cdot \mathbf{v} = \frac{\hbar}{M} \oint \mathrm{d}\mathbf{l} \cdot \nabla \Phi = 2\pi j \frac{\hbar}{M}, \qquad j \in \mathbb{Z}$
- Current conservation: $\frac{\mathrm{d}n}{\mathrm{d}t} + \nabla \cdot (n\mathbf{v}) = 0$
- Thomas-Fermi regime ➜ constant n (*incompressible):* $\nabla \cdot \mathbf{v} = 0$
- A TF superfluid is irrotational & incompressible ➞ a **perfect fluid!**

[A. Fetter, Rev. Mod. Phys. **81**, 647 (2009)]

2D potential flow

stream function

- For 2D incompressible fluids, $\mathbf{v} = \left(\frac{\hbar}{M}\right) \hat{\mathbf{n}} \times \nabla \chi$



- The velocity is parallel to iso-contours of χ
  and orthogonal to iso-contours of Φ

- Perfect fluids in 2D fully described by a *complex potential* $F = \chi + i\Phi$

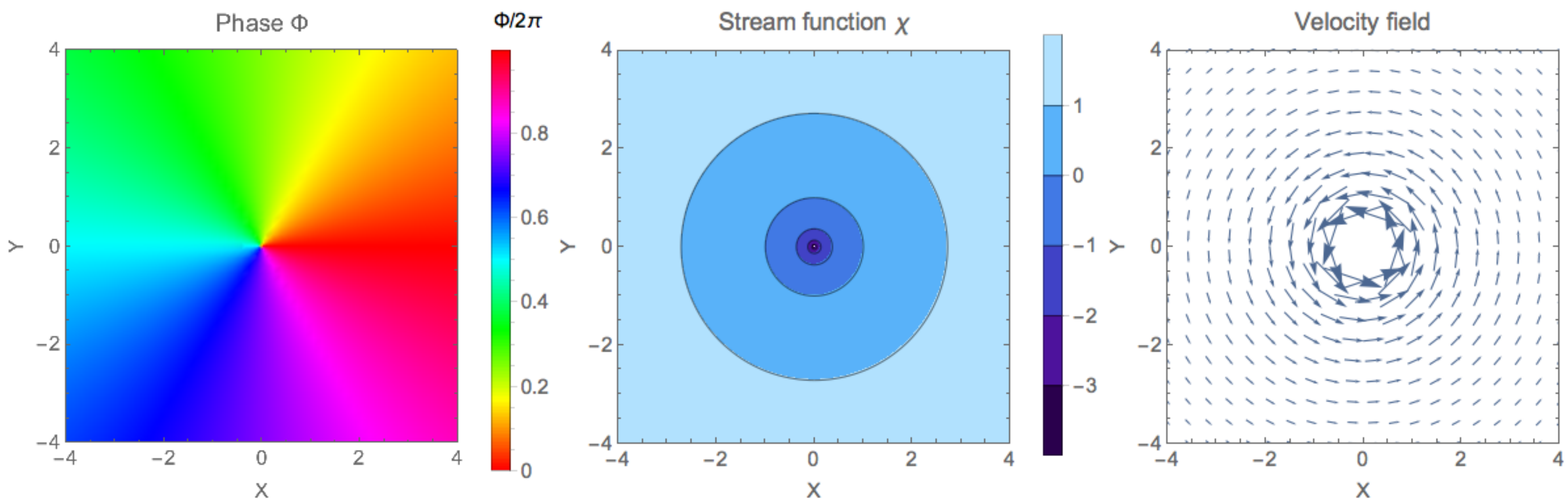
- F is a mero-morphic function of $Z = X + iY$

$$v_X \propto \frac{\partial \Phi}{\partial X} = -\frac{\partial \chi}{\partial Y}$$
$$v_y \propto \frac{\partial \Phi}{\partial Y} = \frac{\partial \chi}{\partial X}$$

- Cauchy-Riemann conditions readily imply: $v_Y + i v_X = \frac{\hbar}{M}\frac{\partial F}{\partial Z}$

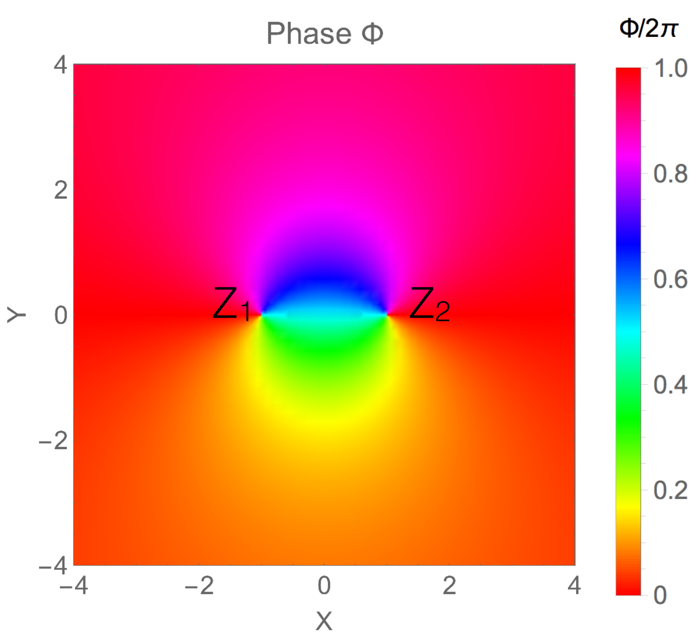
Vortices on a plane

- A single vortex at Z_0 : $F(Z) = \log(Z - Z_0)$



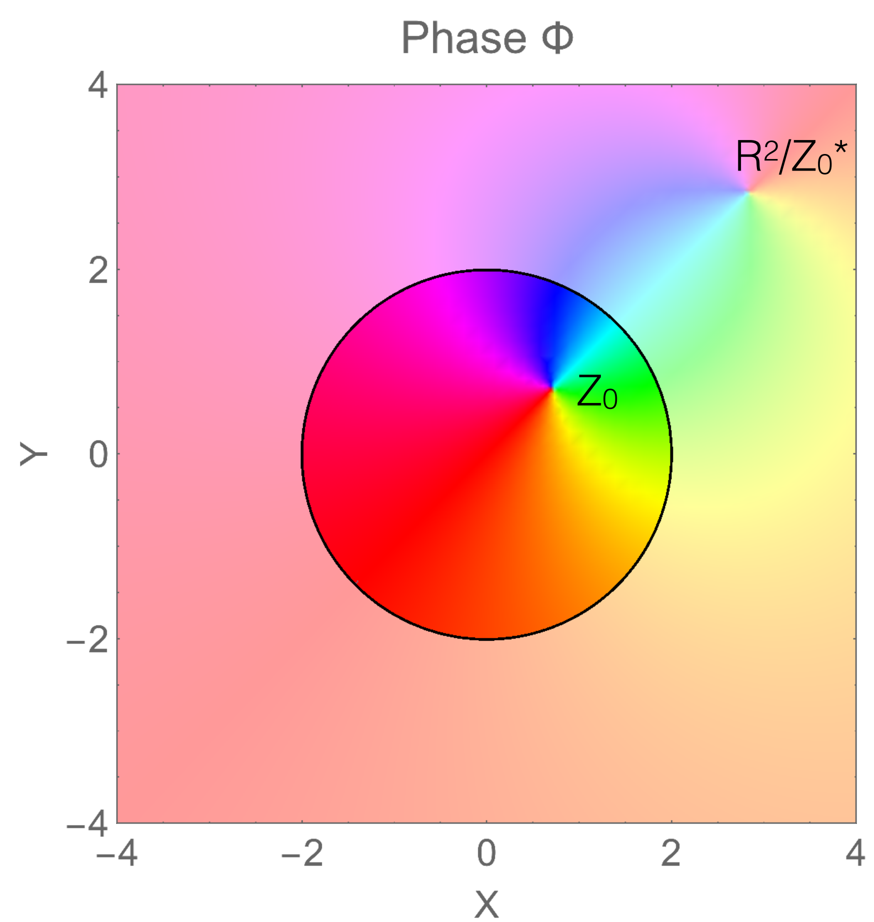
- A vortex dipole:

$$F(Z) = \log(Z - Z_1) - \log(Z - Z_2)$$



Surface with boundaries

- As in electrodynamics, use the method of images
- Single vortex on a disk of radius R: $F(Z) = \log\left(\frac{Z - Z_0}{Z - R^2/Z_0^*}\right)$



Vortex on an annulus

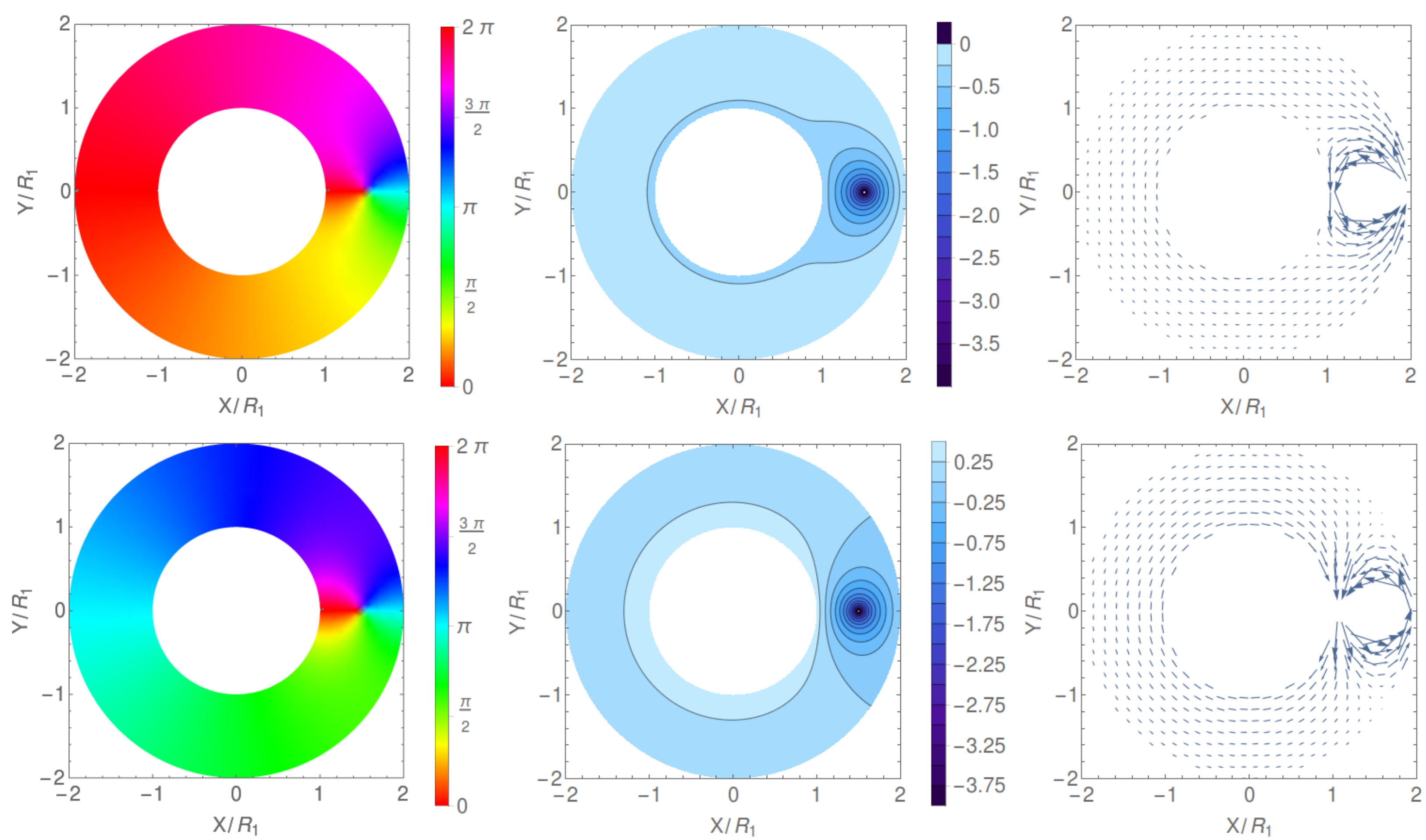
- An annulus has two boundaries → infinite series of images needed

- Potential: $F(Z) = n_1 \ln\left(\frac{Z}{R_2}\right) + \ln\left[\frac{\vartheta_1\left(-\frac{i}{2}\ln\left(\frac{Z}{Z_0}\right), \frac{R_1}{R_2}\right)}{\vartheta_1\left(-\frac{i}{2}\ln\left(\frac{ZZ_0^*}{R_2^2}\right), \frac{R_1}{R_2}\right)}\right]$

1[st] Jacobi Theta function

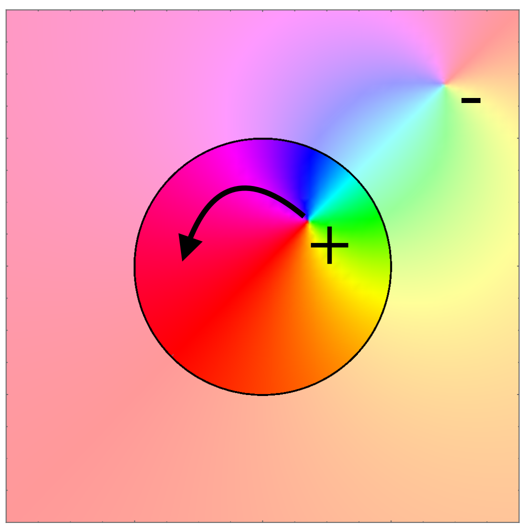
$n_1 = 0$

$n_1 = -1$

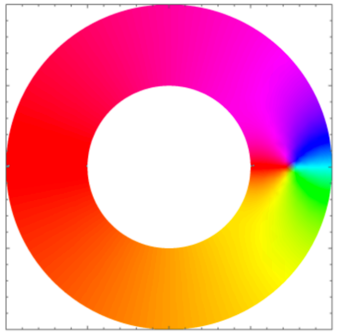


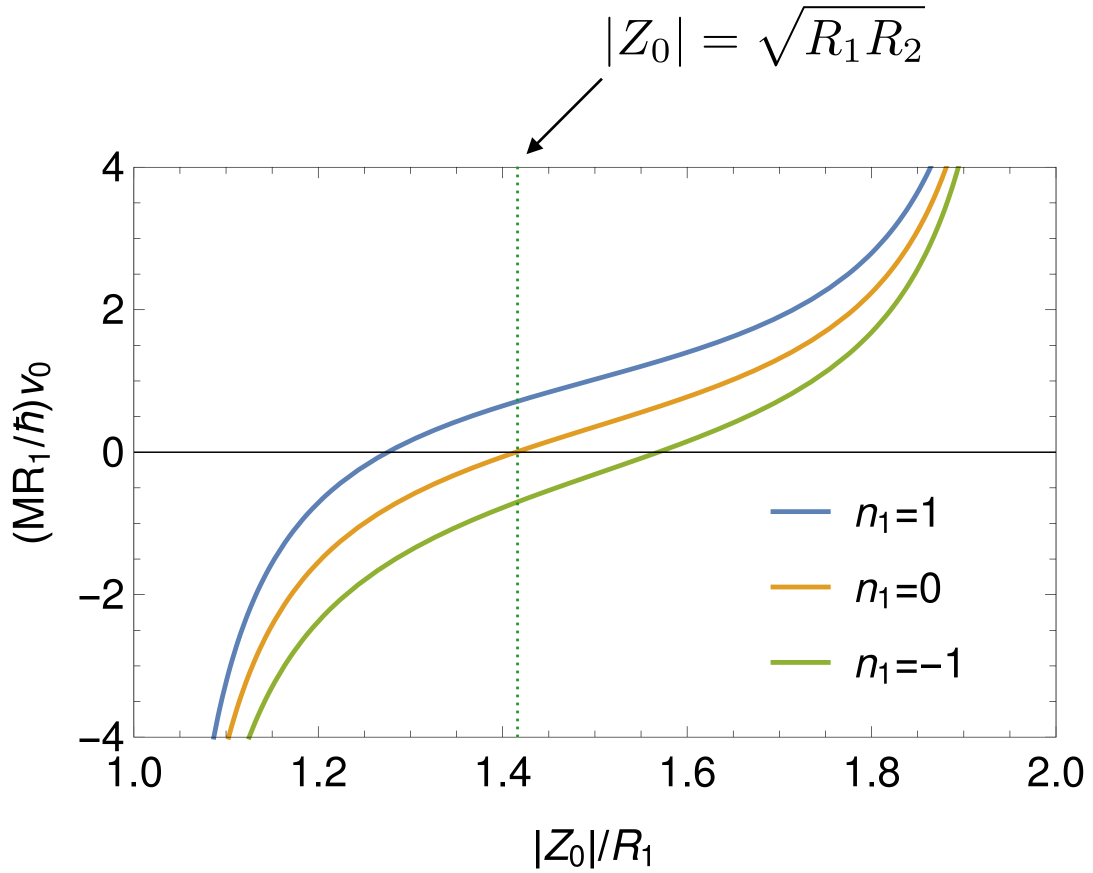
Velocity of the vortex core

- A vortex moves with the local flow velocity: $\dot{y}_0 + i\dot{x}_0 = \frac{\hbar}{M} \lim_{z \to z_0} \left[F'(z) - \frac{1}{z - z_0} \right]$



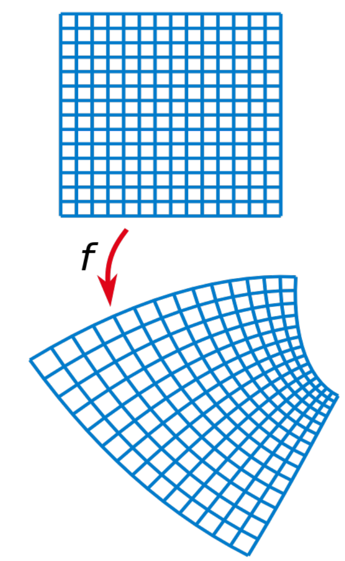
- Annulus with $R_2 = 2R_1$:



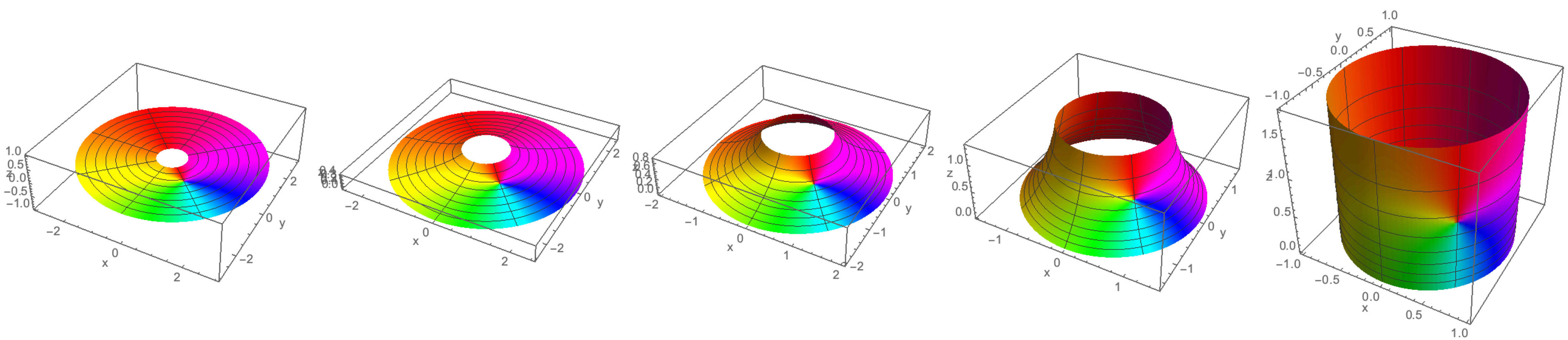


More complex surfaces?

- Conformal map: $f : U \to V$ conserving angles, and shapes of infinitesimal objects

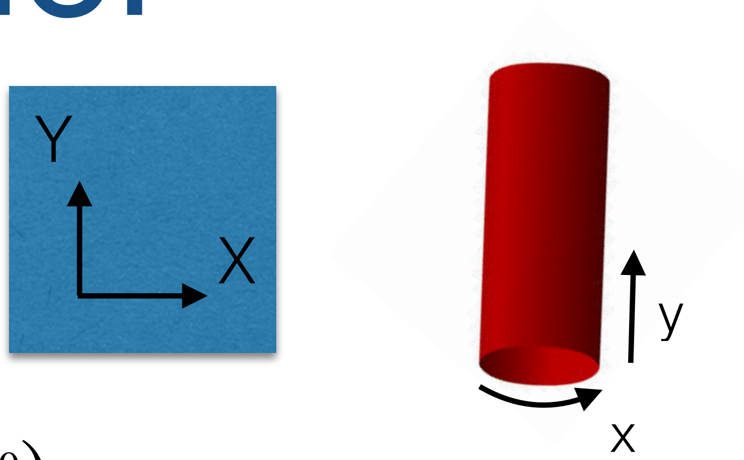


- The conformal image of a physical flow pattern is still a physical pattern



[Turner, Vitelli and Nelson, Rev. Mod. Phys. **82**, 1301 (2010)]

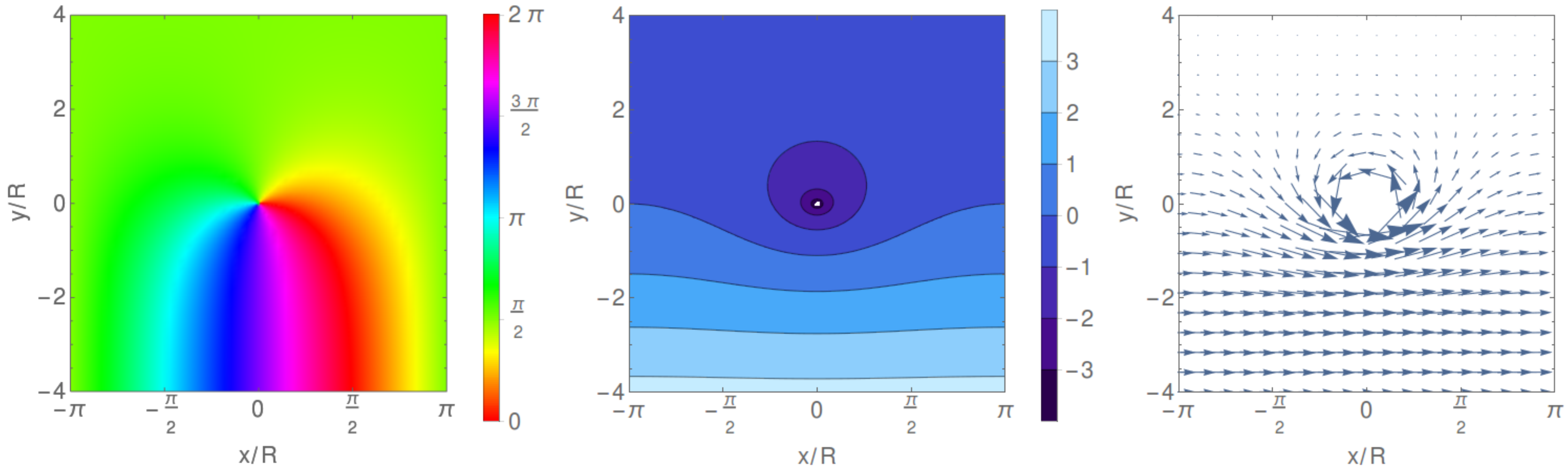
Vortex on a cylinder



- Map linking plane to cylinder: $Z = e^{iz}$

- $F_{\text{plane}} = \log(Z - Z_0) \quad \rightarrow \quad F_{\text{cyl}} = \log(e^{iz} - e^{iz_0})$

F_{cyl} :

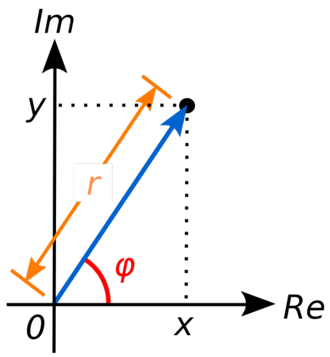


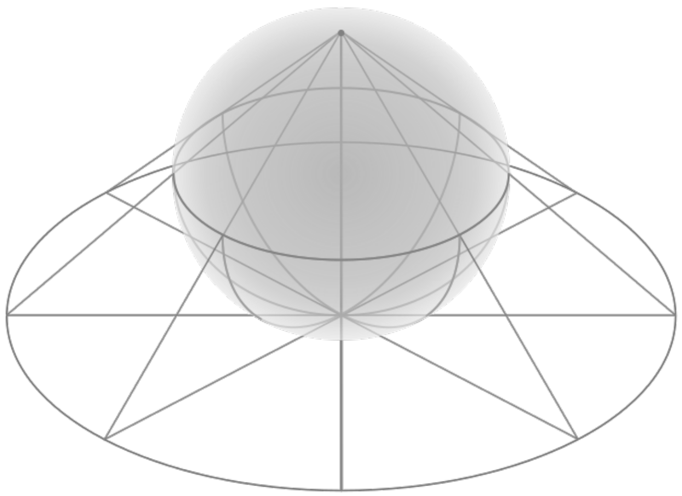
- Velocity of the vortex core: $v_x = \dfrac{\hbar}{2MR}$

a vortex on an infinite cylinder can *not* stand still

Vortices on a sphere

- Stereographic projection:



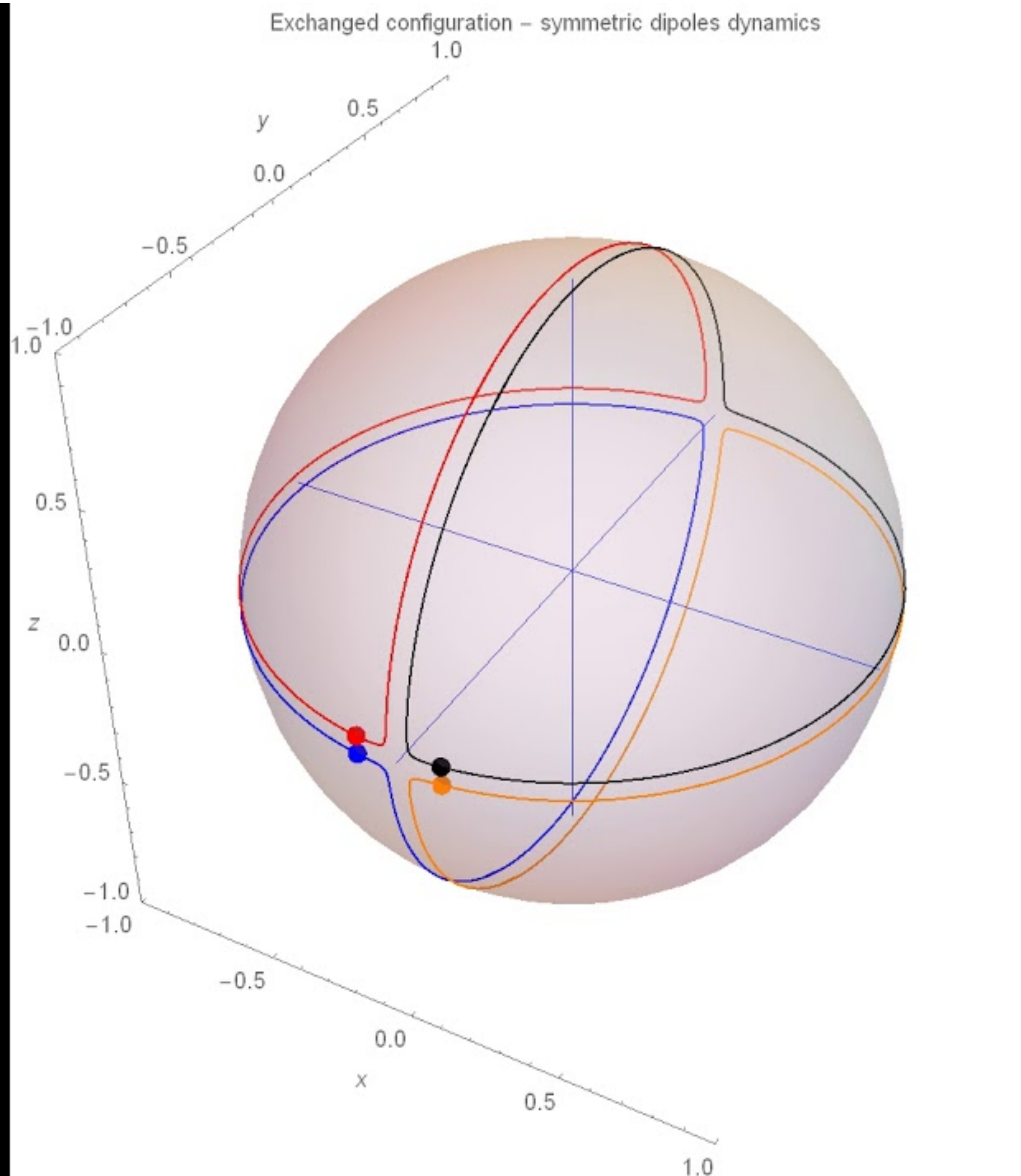


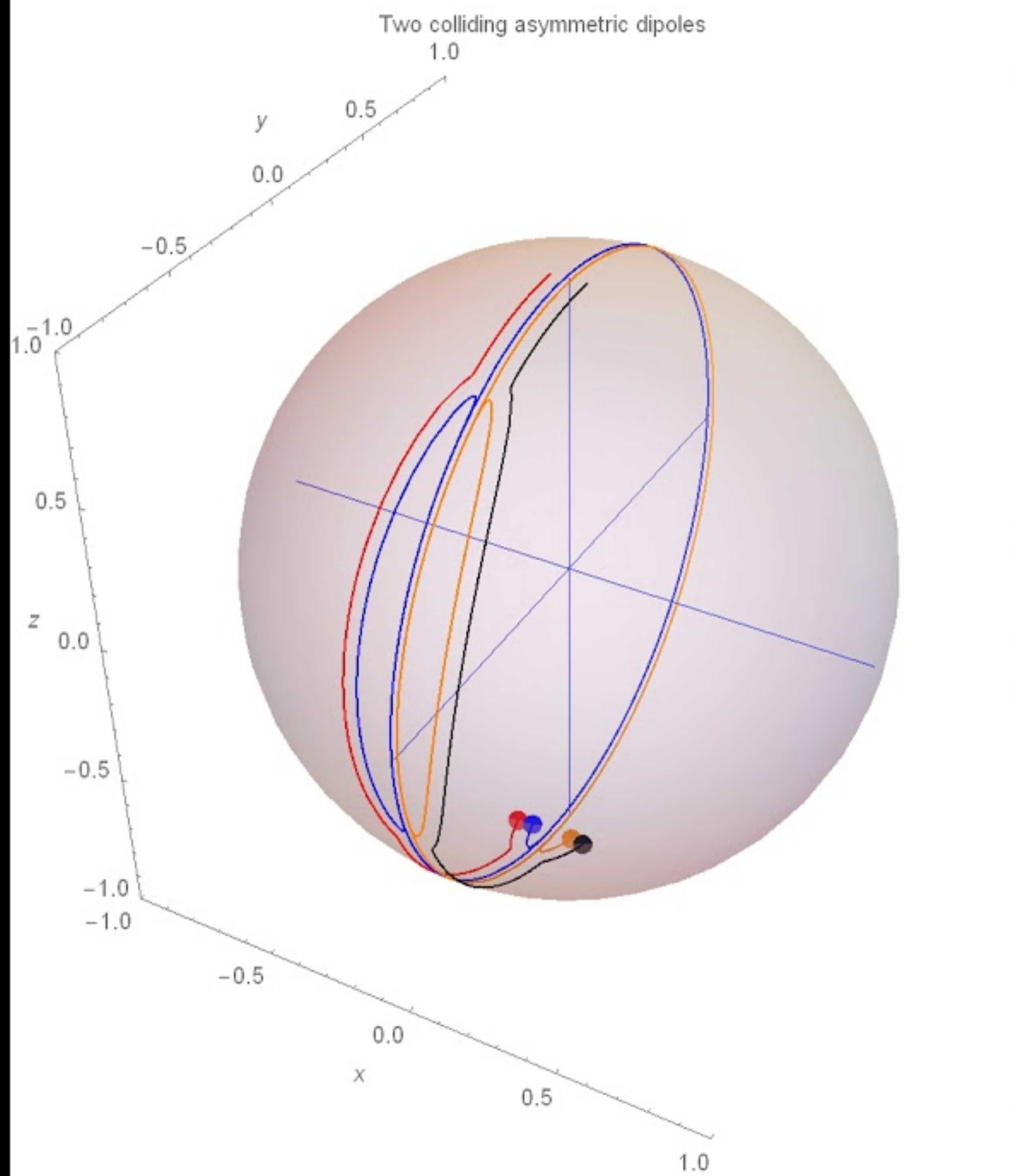
- $F_{\text{plane}} = \log(re^{i\varphi} - Z_0) \quad \rightarrow \quad F_{\text{sph}} = \log[\tan(\theta/2)e^{i\varphi} - \tan(\theta_0/2)e^{i\varphi_0}]$

- Net charge = 0 required on every compact surface!

Lamb, *Hydrodynamics* **(1895)**

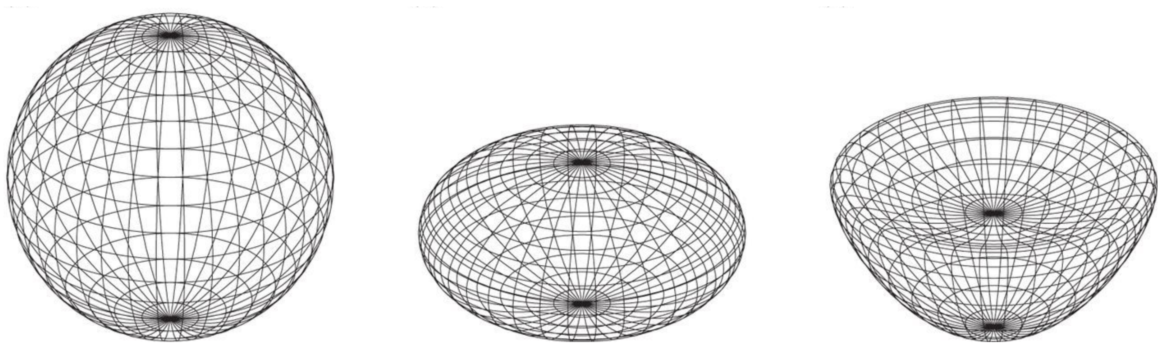
Bereta, Caracanhas and Fetter, PRA (2021)

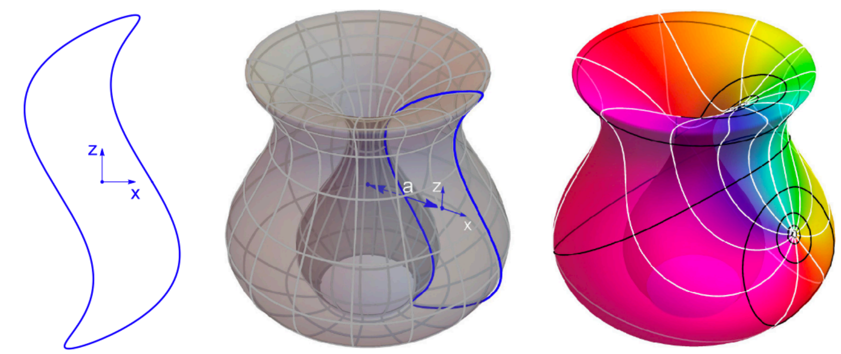

Exchanged configuration – symmetric dipoles dynamics
y
x
z
1.0
0.5
0.0
−0.5
−1.0


Two colliding asymmetric dipoles
y
x
z
1.0
0.5
0.0
-0.5
-1.0

Surfaces of revolution

- $x = f_1(\theta)\cos\phi, \quad y = f_1(\theta)\sin\phi, \quad z = f_2(\theta)$





- Metric on the surface: $ds^2 = dx^2 + dy^2 + dz^2 = h_\theta^2 d\theta^2 + h_\phi^2 d\phi^2$

not isotropic!

- *Isothermal coords.* s.t. $ds^2 = \lambda^2(du^2 + dv^2) = \lambda^2(d\rho^2 + \rho^2 d\phi^2)$

$$\ln\rho(\theta) = \int^{\theta} d\tilde{\theta}\,\frac{h_\theta}{h_\phi}$$

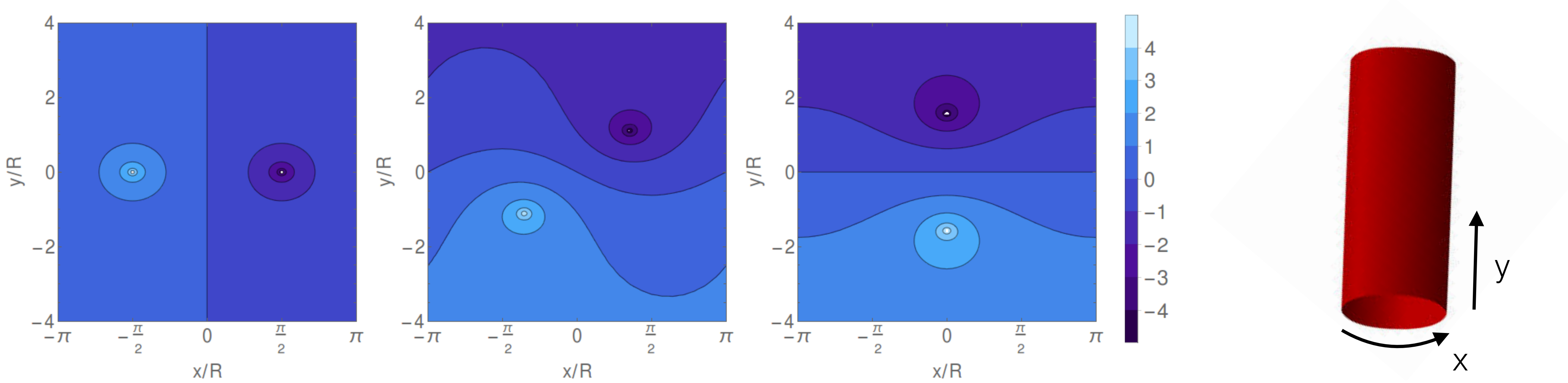
$$\lambda(\theta) = \frac{h_\phi(\theta)}{\rho(\theta)}$$

Kirchhoff, Monatsber. Akad. Wiss. Berlin **(1875)**
Über die stationären elektrischen Strömungen in einer gekrümmten leitenden Fläche

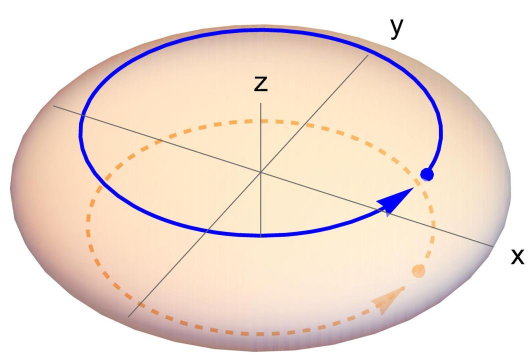
Guenther, Massignan and Fetter, PRA (2020) [torii & genus-1 surfaces]
Caracanhas, Massignan and Fetter, PRA (2022) [ellipsoids & genus-0 surfaces]

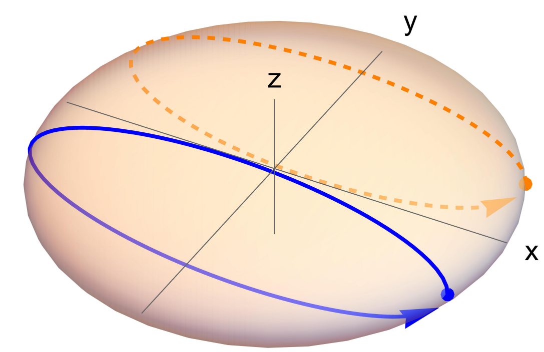
Motion of a vortex dipole

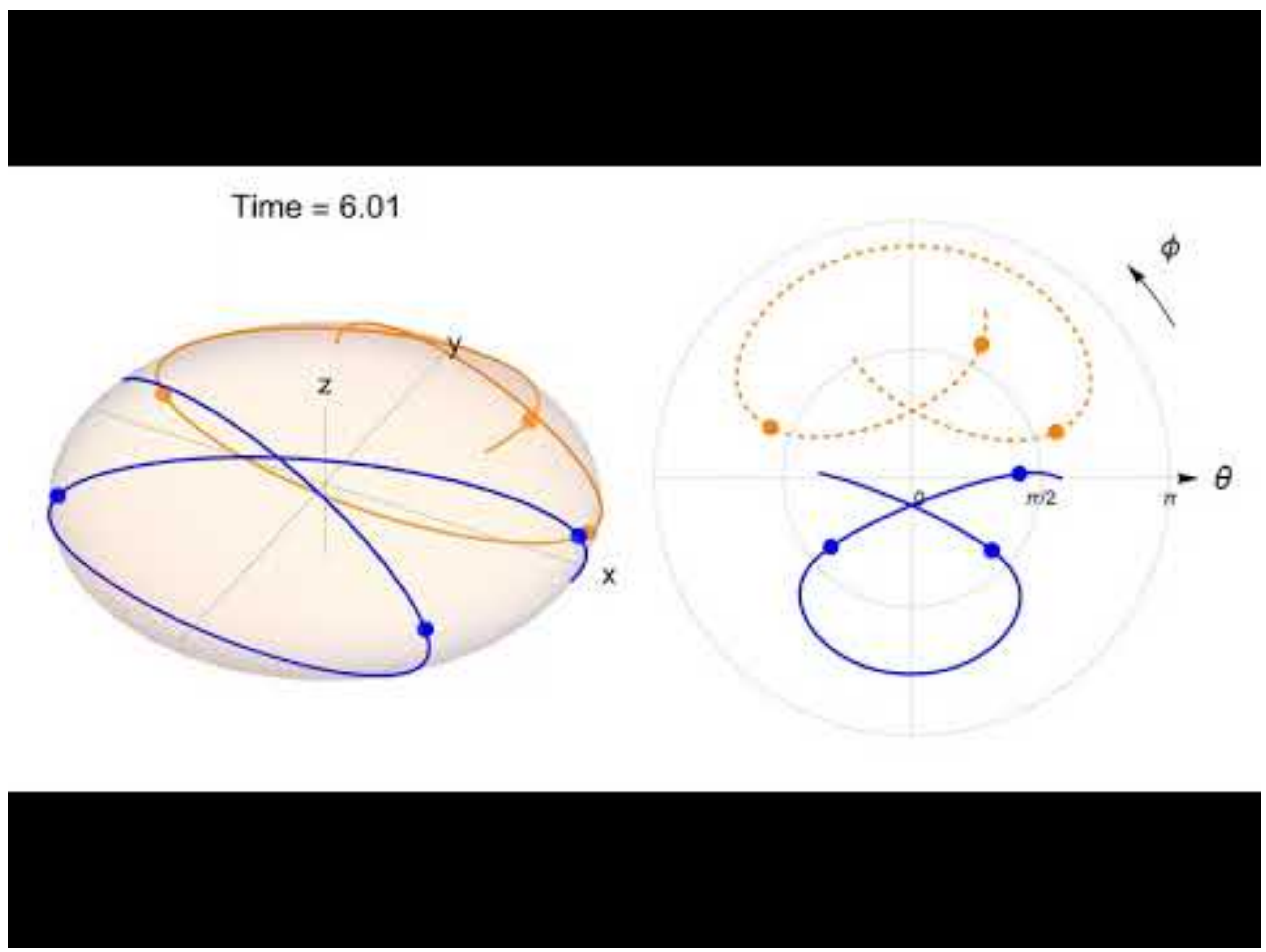
✦ Different trajectories, depending on the orientation of the dipole axis:



Motion of a vortex dipole



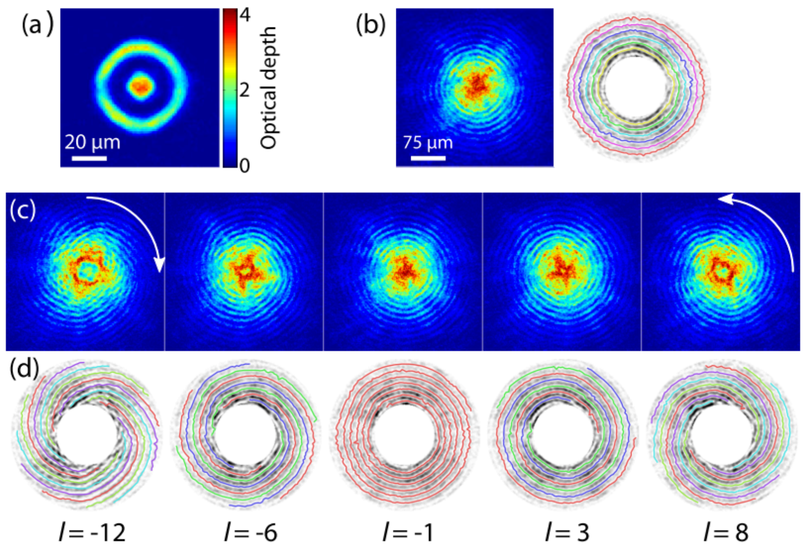




vortices slow down in the regions of higher curvature

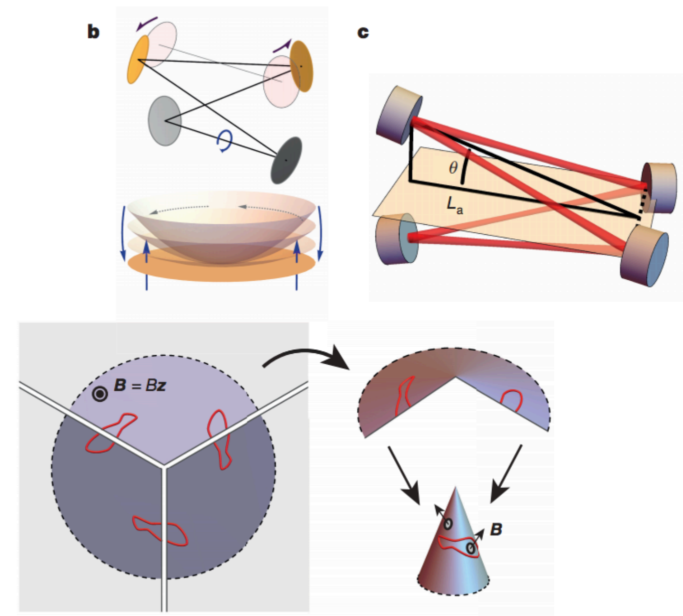
Experiments

Ring traps for BECs:



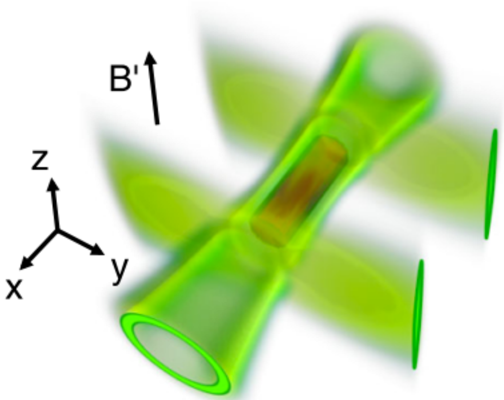
Eckel *et al.*
PRX (2014)

Twisted optical cavities:



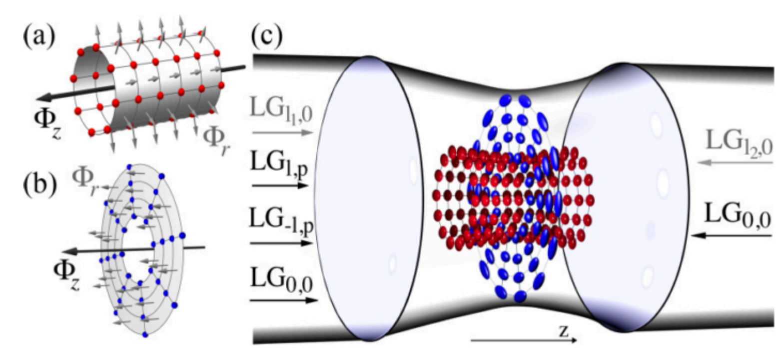
Schine *et al.*
Nature (2016)

Cylindrical traps for BECs:



Gaunt *et al.*
PRL (2013)

Cylindrical and annular lattices for BECs:



Łacki *et al.*
PRA (2016)

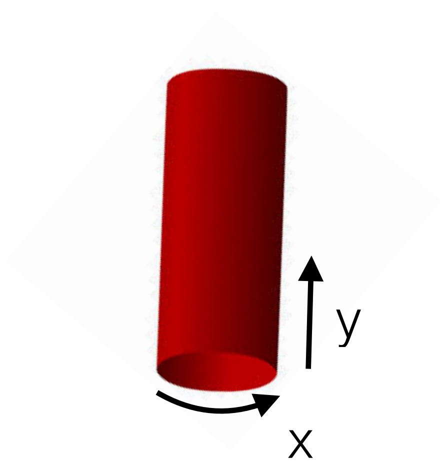
And many **more quantum bubbles** are coming up now!

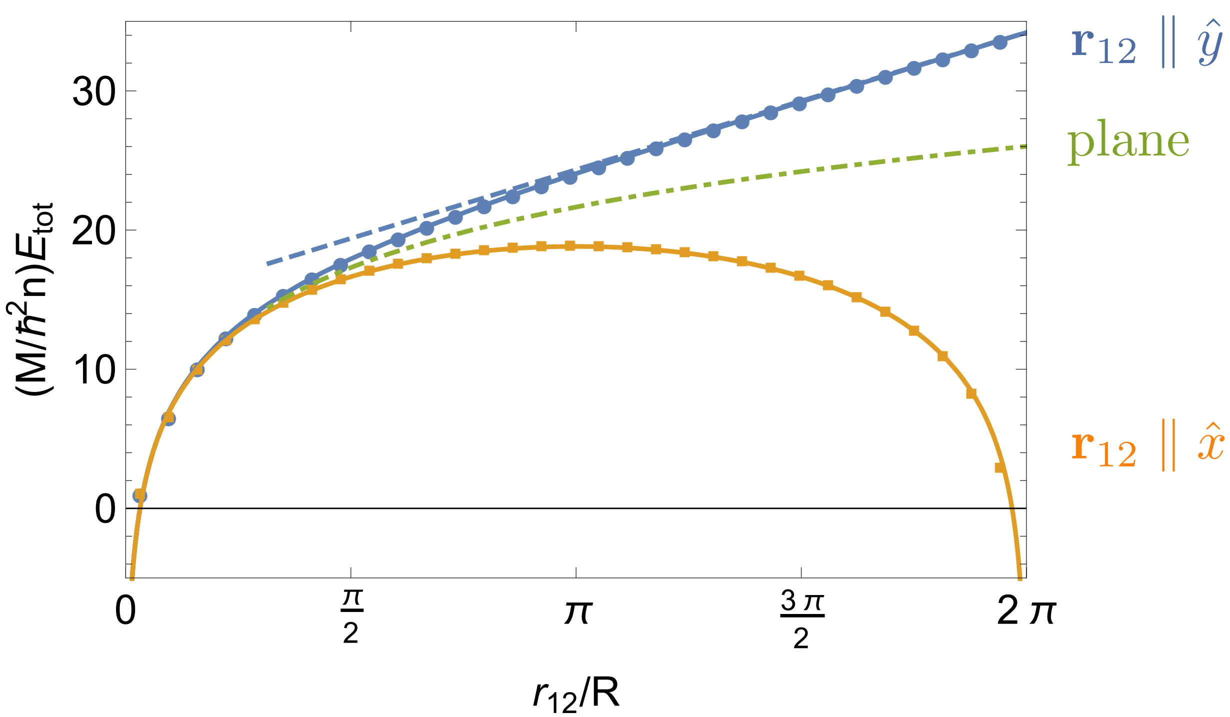
Energy of the fluid

- Energy of N vortices:

$$E = \frac{nM}{2}\int \mathrm{d}^2 r\,|\mathbf{v}(\mathbf{r})|^2 = \frac{\hbar^2 n\pi}{M}\left(-\sum_{kl}{}' q_k q_l\,\chi_{kl} + \sum_k q_k^2 \ln\lambda_k\right)$$

- Energy of a vortex dipole on a cylinder grows linearly for $r_{12} \gg R$

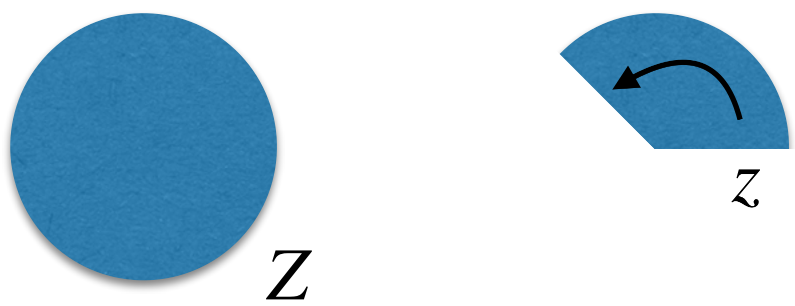


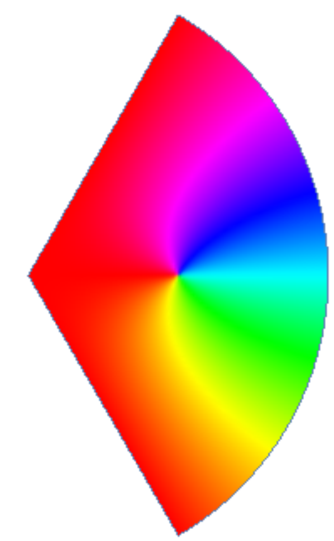


BKT physics will *not* happen on a cylinder

Sectors and cones

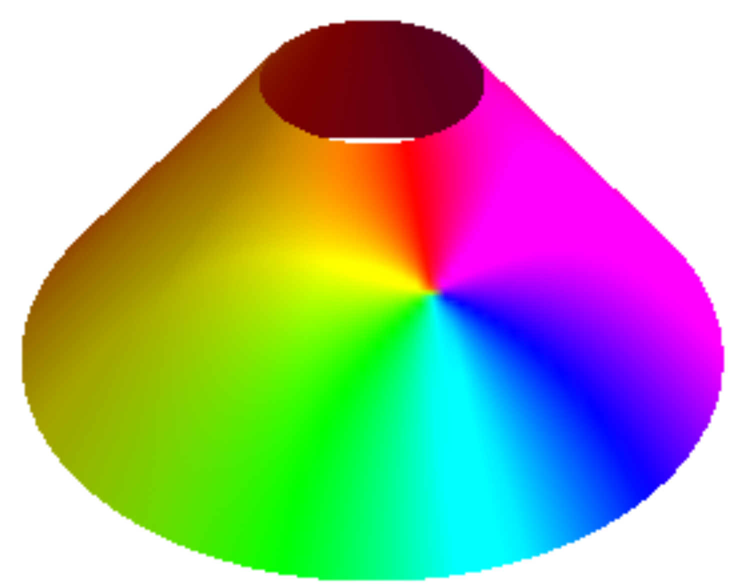
- Conformal map from plane to sector of aperture $2\pi/\alpha$: $Z = z^{\alpha} \quad (\alpha > 1)$





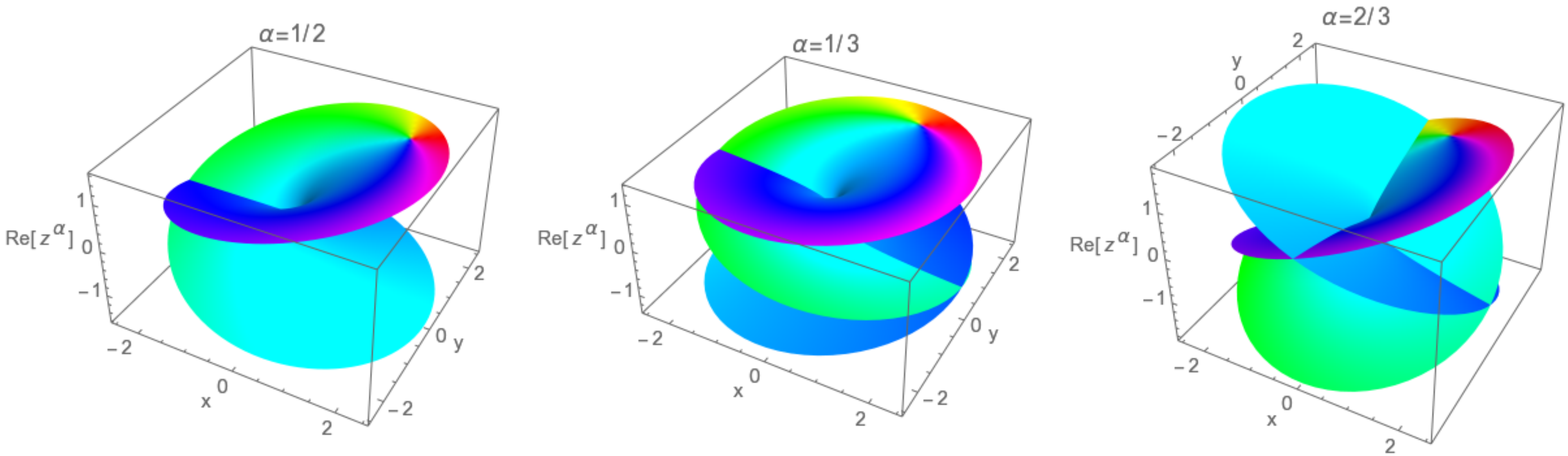
- The sector may be wrapped onto a cone with opening angle $\theta_0 = \arcsin(1/\alpha)$

- A sector of an annulus maps onto a truncated cone:

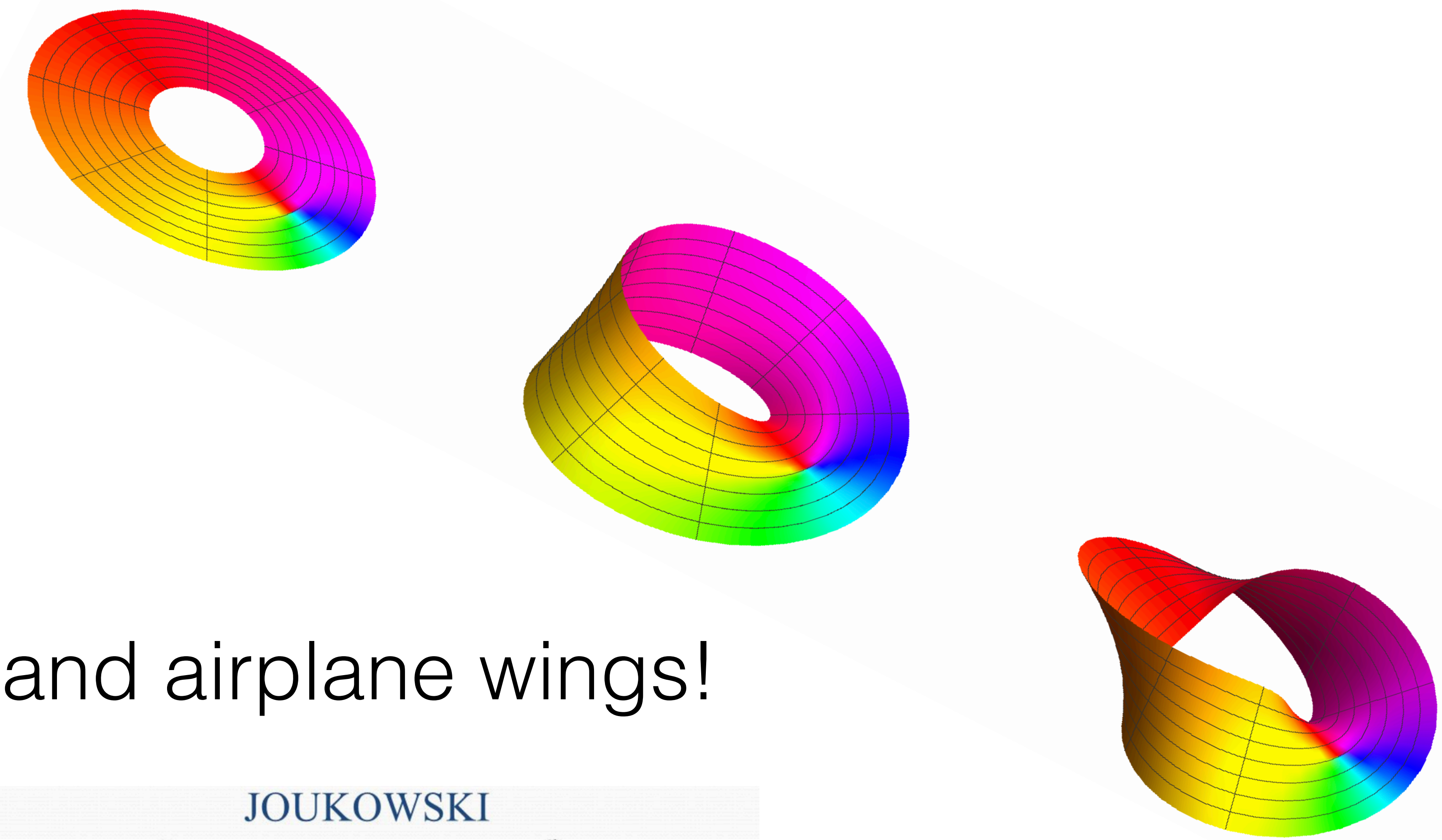


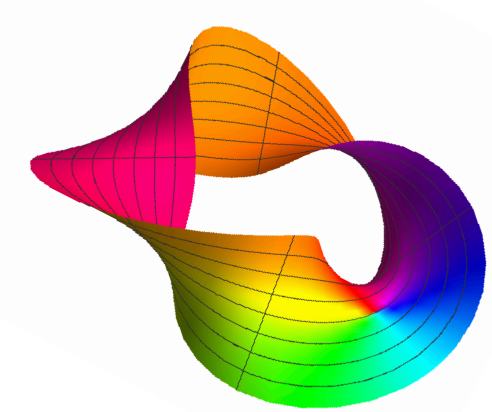
Riemann surfaces

- The map $Z = z^\alpha$ is single-valued only for integer $|\alpha| \geq 1$
- General α ? $0 < \alpha < 1$?
- The function $\text{Re}(z^{p/q})$ winds q times around the origin:

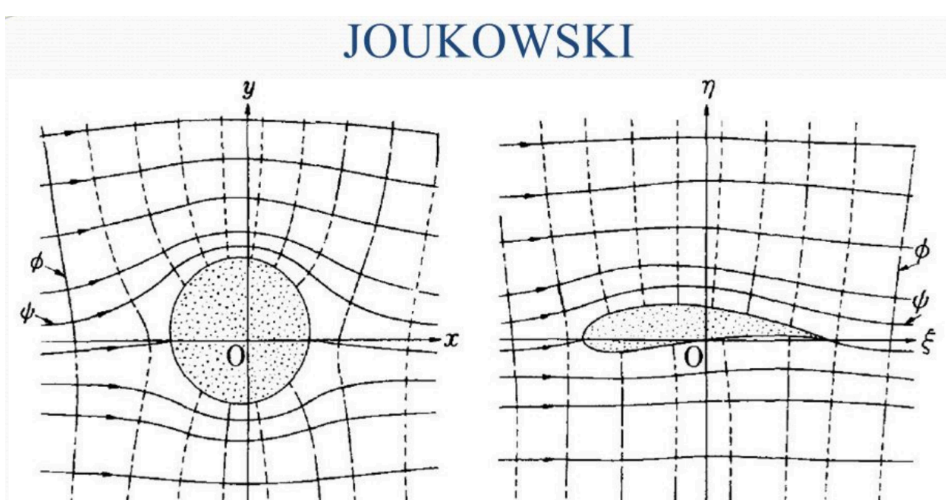


Having fun with Moebius strips

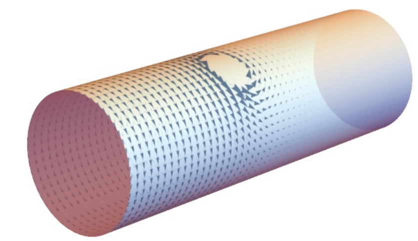




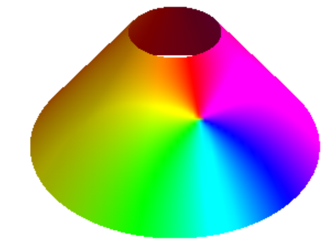
and airplane wings!



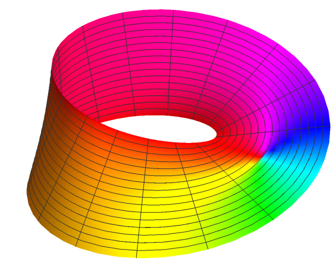
Conclusions

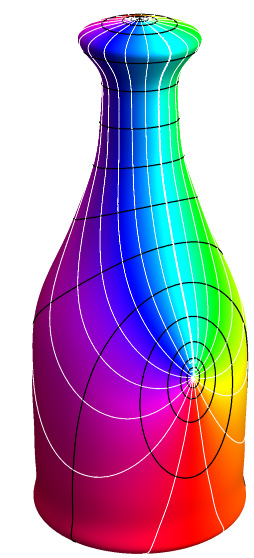


- *Potential flow theory* describes perfect fluids in 2D.

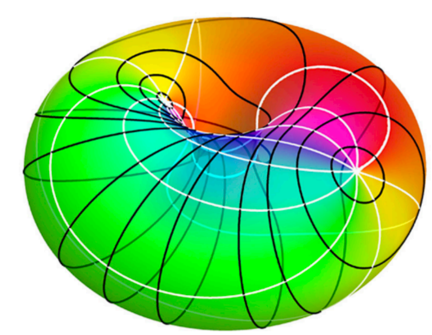


- Image charges, conformal maps and isothermal coordinates permit the study of peculiar geometries.

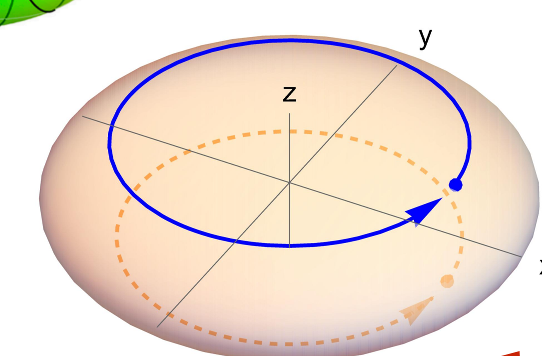




- Single-valuedness of Ψ around closed loops
  → quantized translational velocities of vortex cores.
- On cylinders and torii, vortices will not stand still.



- Vortices slow down in the regions of higher curvature.
- BKT physics will not happen on a cylinder.



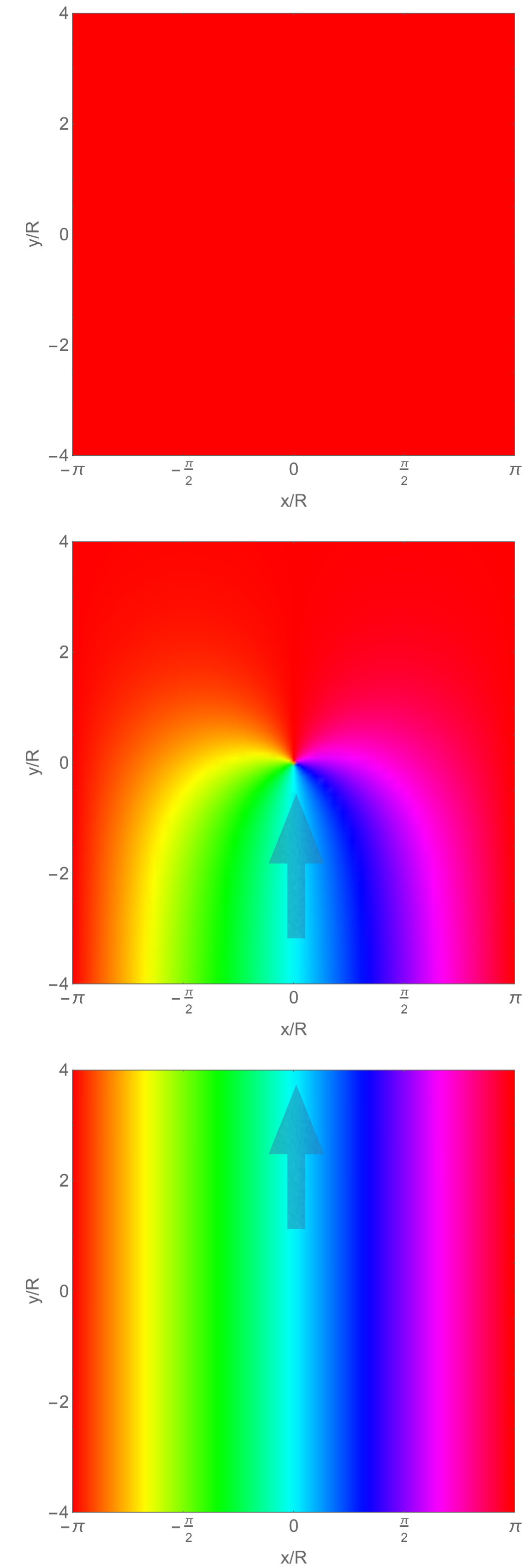
Cylinders and annuli: Guenther, PM and Fetter, PRA (2017)
Cones and sectors: PM and Fetter, PRA (2019)
Torii and genus-1 surfaces: Guenther, PM and Fetter, PRA (2020)
Ellipsoids and genus-0 surfaces: Caracanhas, PM and Fetter, PRA (2022)

BEC on a sphere: Tononi and Salasnich, PRL (2019)
GP simulations of vortices on torii: D'Ambroise *et al.*, arXiv:2201.11054

Thank you!

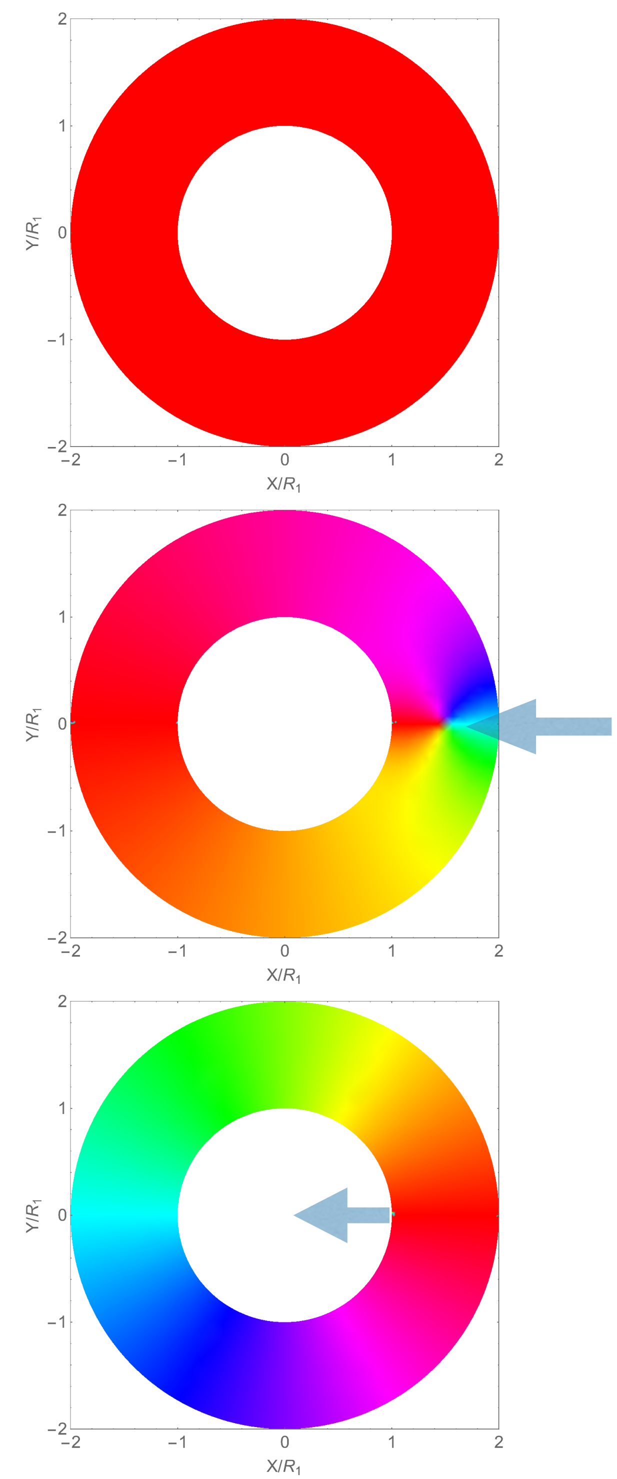
Laughlin pumping on a cylinder

- Start with the fluid at rest
- Stir the fluid from below at a constant rate
- A vortex appears on the lower edge, and moves upward
- The fluid remains stationary above the vortex
- Below the vortex, the fluid rotates with exactly $\hbar$ units of angular momentum per particle

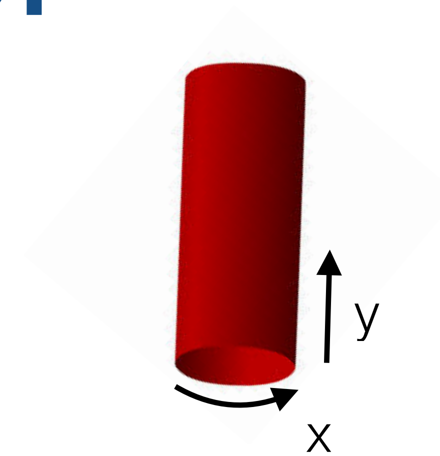


Laughlin pumping on an annulus

- Start with the fluid at rest
- Stir the fluid from outside at a constant rate
- A vortex appears on the outer edge, and moves inward
- The fluid (on average) rotates for $|Z| > |Z_0|$, but it remains stationary otherwise
- As the vortex crosses the inner edge, stop stirring
- The fluid is left with exactly $\hbar$ units of angular momentum per particle



Vortex on a finite cylinder



- Cylinder of length L: $$F_L(z) = \ln\left[\frac{\vartheta_1\left(\frac{z-z_0}{2R}, e^{-L/R}\right)}{\vartheta_1\left(\frac{z-z_0^*}{2R}, e^{-L/R}\right)}\right]$$

- Vortex velocity: